1–6 Mathematical Induction

Suppose that the following claim is made: For all natural numbers, n, the natural number $n^3 + 2n$ is divisible by 3. Is the claim true? Is it false? To find out you might begin by examining several cases.

$n = 1$: $1^3 + 2 \cdot 1 = 3$; 3 is divisible by 3.
$n = 2$: $2^3 + 2 \cdot 2 = 12$; 12 is divisible by 3.
$n = 3$: $3^3 + 2 \cdot 3 = 33$; 33 is divisible by 3.

So far so good. Now check the statement for $n = 4$, $n = 5$, and $n = 6$. What are your results?

One thing is clear: You cannot possibly substantiate the claim for all natural numbers n. Only a finite number of instances could be checked. What you need, then, is a method of proof which verifies the claim for all natural numbers at once. The method of proof sought is **mathematical induction.**

Mathematical induction is based on simple characteristics of the natural numbers. They are stated in the **Axiom of Induction.**

Axiom of Induction: If T is a set of natural numbers with the properties

i. $1 \in T$ and
ii. $k \in T$ implies $k + 1 \in T$,

then T is the set of natural numbers N.

The pair of conditions in the Axiom of Induction uniquely determine the set of natural numbers. Notice that both conditions must be satisfied for the Axiom of Induction to hold. The following example will help to illustrate the point.

For example, consider the following set T.

$$T = \{1, 2, 3\}$$

Clearly $1 \in T$, but it is not true that $k + 1 \in T$ every time $k \in T$, e.g., $3 \in T$ but $4 \notin T$. Thus $T \neq N$.

$$\text{Let } T = \{10, 11, 12, \cdots\}.$$

Clearly if $k \in T$, then $k + 1 \in T$ but $1 \notin T$. Thus, again, $T \neq N$.

The Theorem of Mathematical Induction follows directly.

Theorem 1–7 For every $n \in N$ let P_n be a statement which is either true or false. If

i. P_1 is true and
ii. whenever this statement is true for k, it is true for $k + 1$

then P_n is true for all $n \in N$.

Proof: Let T be the set of natural numbers for which P_n is true.

$1 \in T$ because of i in Theorem 1–7.
$k + 1 \in T$ whenever $k \in T$ because of ii.

Thus $T = N$ and P_n is true for all $n \in N$.

To apply the Theorem of Mathematical Induction you must do *two* things: You must verify that P_1 is true. You must also verify that P_{k+1} is true whenever P_k is true.

EXAMPLE 1. For n a natural number, P_n is the statement:

$$n^3 + 2n \text{ is divisible by 3.}$$

Prove P_n true for all $n \in N$.

The proof is by mathematical induction.

i. *Verify that* P_1 *is true.*

$$P_1 : 1^3 + 2 \cdot 1 \text{ is divisible by 3.}$$

P_1 is clearly true since $1^3 + 2 \cdot 1 = 3$.

ii. *Verify that whenever* P_n *is true for a natural number, say* $n = k$, *then it is true for the next natural number,* $n = k + 1$.

To carry out this portion of the argument, you must assume that P_k is true for some $k \in N$. This assumption is the *induction hypothesis.*

Assume P_k is true; i.e., $k^3 + 2k$ is divisible by 3. Next show that P_k implies P_{k+1}, i.e., prove that P_{k+1} must also be true; i.e., $(k+1)^3 + 2(k+1)$ is divisible by 3. First notice that

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2\\ &= (k^3 + 2k) + 3k^2 + 3k + 3\\ &= P(k) + 3(k^2 + k + 1)\end{aligned}$$

The symbol $P(k)$ is used in the last line to stand for "$k^3 + 2k$". It should not be confused with the symbol P_k which stands for the sentence "$k^3 + 2k$ is divisible by 3". Similarly, $P(k+1)$ stands for "$(k+1)^3 + 2(k+1)$".

Since $P(k)$ is divisible by 3 and $3(k^2 + k + 1)$ is divisible by 3, $P(k+1)$ is also divisible by 3. Thus both conditions of the Theorem of Mathematical Induction are satisfied. You may thus conclude that $n^3 + 2n$ is divisible by 3 for *all* $n \in N$.

EXAMPLE 2. For n a natural number, P_n is the statement:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Prove P_n true for all $n \in N$.

i. P_1 is true, since $\frac{1(1+1)}{2} = 1$.

ii. Induction hypothesis:

Assume P_k is true; i.e., $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.

Prove that P_{k+1} follows from P_k.

The left hand side of P_{k+1} is $1 + 2 + 3 + \cdots + k + (k+1)$. But this is the left side of P_k with $k+1$ added. This suggests that a proof may be made by adding $(k+1)$ to both sides of the true statement P_k.

Thus
$$\begin{aligned}1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k(k+1) + 2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\\ 1 + 2 + \cdots + k + (k+1) &= \frac{(k+1)(k+1+1)}{2}\end{aligned}$$

The last statement is P_{k+1}. Thus P_{k+1} follows from P_k.

By the Theorem of Mathematical Induction P_n is true for all $n \in N$.

EXAMPLE 3. For n a natural number, P_n is the statement:

$$(1+p)^n \geq 1 + np, \quad p > -1.$$

Prove P_n true for all $n \in N$.

i. Verify P_1.

$$P_1 : (1+p)^1 \geq 1 + 1 \cdot p$$
$$1 + p \geq 1 + p, \quad p > -1$$

Thus P_1 is true.

ii. Induction Hypothesis.

Assume P_k is true, i.e., $(1+p)^k \geq 1 + kp$. Now prove that the truth of P_{k+1} follows from that of P_k. The left hand side of the statement P_{k+1} is $(1+p)^{k+1} = (1+p)(1+p)^k$. The second factor in $(1+p)(1+p)^k$ is the left hand side of P_k. This suggests that you may be able to make the proof by beginning with P_k and multiplying by $(1+p)$. You know that

$$(1+p)^k \geq 1 + kp.$$

Since, $1 + p \geq 0$ then

$$(1+p)(1+p)^k \geq (1+kp)(1+p)$$

or
$$(1+p)^{k+1} \geq 1 + kp + p + kp^2 = 1 + (k+1)p + kp^2.$$

Since $kp^2 \geq 0$

$$1 + (k+1)p + kp^2 \geq 1 + (k+1)p$$

Thus
$$(1+p)^{k+1} \geq 1 + (k+1)p + kp^2 \geq 1 + (k+1)p$$

or
$$(1+p)^{k+1} \geq 1 + (k+1)p$$

But this is P_{k+1}! Thus the statement P_n is true for all $n \in N$.

It is important to realize that both parts of the Theorem of Mathematical Induction must be satisfied for P_n to be true for all natural numbers n. For example, suppose P_n is the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} + \frac{(n-1)}{2}$$

P_1 is true since "$1 = \frac{1(1+1)}{2} + \frac{(1-1)}{2}$" is true.

In this case, however, you cannot conclude that whenever P_k is true, P_{k+1} is also true because there is no general way to produce P_{k+1} from P_k. Thus P_n is not true for all $n \in N$. The way to disprove this statement is to produce a counterexample, i.e., a natural number for which P_n is not true. P_3 is not true because

$$6 = 1 + 2 + 3 \neq \frac{3(3 + 1)}{2} + \frac{(3 - 1)}{2} = 6 + 1 = 7.$$

On the other hand suppose that P_n is the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} + 17.$$

Then you can show that *if* P_k is true, P_{k+1} must be true also. However, it is impossible to show that P_n is true for *any* n much less for $n = 1$.

Exercises

A In Exercises 1–12 P_n is given. Prove by Mathematical Induction that P_n is true for all $n \in N$.

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

2. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

3. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$

4. $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

5. $-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^n} = \frac{1 - (2)^n}{2^n}$

6. $(b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}$

7. $5 + 5 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3^2} + \cdots + 5 \frac{1}{3^{n-1}} = 5 \frac{1 - (\frac{1}{3})^n}{1 - \frac{1}{3}}$

8. $5 + 7 + 9 + \cdots + [5 + 2(n - 1)] = \frac{n(10 + 2n - 2)}{2}$

B

9. $a + aq^1 + aq^2 + \cdots + aq^{n-1} = a\frac{1 - q^n}{1 - q}$ $a, q \in R, q \neq 1$

10. $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n[(2a + (n - 1)d]}{2}$ $a, d \in R$

11. $n^3 - n$ is divisible by 6

12. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

13. Prove that the sum of n positive integers is positive. (*Hint:* See Section 1–3, Postulate 17.)

14. Prove that the product of n positive integers is positive. (*Hint:* See Section 1–3, Postulate 18.)

15. Prove that if $x_0 < x_1, x_1 < x_2, x_2 < x_3, x_3 < x_4, \ldots, x_{n-1} < x_n$, then $x_0 < x_n$. (*Hint:* Use Theorem 1–4.)